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New integrable models from fusion

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Abstract. Integrable multistate or multicolour models were recently introduced. They are generalizations of models corresponding to the defining representations of the $U_q(\widehat{sl(m)})$ quantum algebras. Here I show that a similar generalization is possible for all higher-dimensional representations. The R -matrices and the Hamiltonians of these models are constructed by fusion. The $sl(2)$ case is treated in some detail and the spin-0 and spin-1 matrices are obtained in explicit forms. This provides in particular a generalization of the Fateev–Zamolodchikov Hamiltonian.

1. Introduction

New one-dimensional integrable lattice models were introduced in [1], within the framework of the quantum inverse scattering method [2–4]. They correspond to a generalization whereby every state of the original model, for the defining representation of the Lie algebra A_{m-1} (see for instance [5]), is replaced by an arbitrary number of copies. The structure of the R -matrices is left unchanged by the replacement, but a usual property, crossing unitarity, is lost. However, it is still possible to construct integrable open boundary conditions. The eigenvalue set of the transfer matrix is unchanged, while the degeneracies increase [6]. For integrable periodic boundary conditions the eigenvalues and degeneracies change [1].

These models are of interest in some one-dimensional reaction–diffusion processes [7]. (Similar but different models were also considered in [8] as generalizations of the t – J model.) They also appear in connection with the Hubbard model for electrons, as a specific limit [9, 10] or, for $m = 2$ and at a certain value of the quantum parameter, as building blocks for the natural generalizations of the Hubbard model [11, 12]. Upon fermionization, the multiple states appear as different flavours of fermions. The general Hubbard models can then be seen as multichannel versions of the original model.

All the ‘multiplicity’ A_{m-1} models of [1] correspond to the *defining* representations of the Lie algebras A_{m-1} . As no quantum group formalism [5] is yet known for the multiplicity models, it is not clear whether generalizations to higher-dimensional representations are possible. There is no obvious or systematic way to perform a replacement of states by multiple copies in higher-dimensional R -matrices. The fusion method, known to work for ordinary models, turns out to give the correct answer for the multistate models.

The main result of this work is to show that such generalizations exist. I first review the fusion method for obtaining higher-dimensional solutions of the Yang–Baxter equation starting from an arbitrary solution. The results of section 2 are quite general and require only a minimal number of assumptions. This method is then shown to work for the models at hand. This yields multiplicity models corresponding to some higher-dimensional representations,

and also shows that iterated fusions are possible. Thus all higher-dimensional representations of $sl(m)$ have multistate generalizations. Hamiltonians are obtained as the derivative of the new \hat{R} -matrices. I consider in detail the spin-0 and spin-1 models of $sl(2)$, and give explicit expressions for their R -matrices.

2. Fusion

The multiple fusion procedure for $sl(m)$ R -matrices was developed in [13]. The idea for other algebras is the same [14, 15]. A simple description was given in [16] for two successive fusions. Here I review the fusion method and give some additional general results.

The models are defined through their R -matrices. These are solutions of the Yang–Baxter equation (YBE)

$$R_{12}(\lambda - \mu) R_{13}(\lambda) R_{23}(\mu) = R_{23}(\mu) R_{13}(\lambda) R_{12}(\lambda - \mu). \quad (1)$$

Here and below, the notation \mathcal{O}_{ij} ($i \neq j$) means that the operator \mathcal{O} acts non-trivially on the i th and j th spaces, and as the identity on the other spaces: $\mathcal{O}_{ij} = \sum_k \mathbb{I} \otimes \cdots \otimes a_k^{(i)} \otimes \cdots \otimes b_k^{(j)} \otimes \cdots \otimes \mathbb{I}$ (if $i < j$), where $\mathcal{O} = \sum_k a_k \otimes b_k$. Note also that the three spaces 1, 2 and 3 need *not* be copies of the same space.

Consider *any* solution R of the Yang–Baxter equation (1), which becomes proportional to a projector at some special value ρ of the spectral parameter. Define the projector $\pi^{(1)}$ so that $\pi^{(1)} \propto R(\rho)$, and let $\pi^{(2)} = \mathbb{I} - \pi^{(1)}$ be the orthogonal complementary projector. Setting $\lambda - \mu = \rho$ in the YBE one obtains

$$\pi_{12}^{(1)} R_{13}(\lambda) R_{23}(\lambda - \rho) = R_{23}(\lambda - \rho) R_{13}(\lambda) \pi_{12}^{(1)}. \quad (2)$$

Left and right multiplication of (2) by $\pi_{12}^{(2)}$ yields two equations

$$\pi_{12}^{(1)} R_{13}(\lambda) R_{23}(\lambda - \rho) \pi_{12}^{(2)} = 0 \quad (3)$$

$$\pi_{12}^{(2)} R_{23}(\lambda - \rho) R_{13}(\lambda) \pi_{12}^{(1)} = 0. \quad (4)$$

Define two fused matrices by

$$R_{(12)3}^{(1)}(\lambda) = \pi_{12}^{(1)} R_{13}(\lambda) R_{23}(\lambda - \rho) \pi_{12}^{(1)} \quad (5)$$

$$R_{(12)3}^{(2)}(\lambda) = \pi_{12}^{(2)} R_{13}(\lambda) R_{23}(\lambda - \rho) \pi_{12}^{(2)}. \quad (6)$$

Using equations (3), (4) and the YBE one shows that the matrices (5)–(6) satisfy a YBE where one space is a tensor product of two spaces:

$$R_{(12)3}^{(i)}(\lambda - \mu) R_{(12)4}^{(i)}(\lambda) R_{34}(\mu) = R_{34}(\mu) R_{(12)4}^{(i)}(\lambda) R_{(12)3}^{(i)}(\lambda - \mu) \quad i = 1, 2. \quad (7)$$

Thus starting with a given solution of the YBE, we have obtained higher-dimensional R -matrices which are also solutions of the YBE. If d_i , $i = 1, 2, 3$ are the dimensions of the spaces 1, 2 and 3, then $R_{(12)3}^{(i)}(\lambda)$ is a $d_1 \times d_2 \times d_3$ dimensional square matrix. However the projection operators can be diagonalized simultaneously with a change of basis matrix S . $S_{12}^{-1} R_{(12)3}^{(i)}(\lambda) S_{12}$ also satisfy equation (7), and their matrix expressions now contain a number of rows and columns with only vanishing elements, and which can be deleted without spoiling the Yang–Baxter property. The remaining rows and columns can eventually be relabelled according to the states of the corresponding representations. Let $\text{tr}(\pi^{(1)}) = d$. Deleting the vanishing rows and columns from $S_{12}^{-1} R_{(12)3}^{(1)}(\lambda) S_{12}$ and $S_{12}^{-1} R_{(12)3}^{(2)}(\lambda) S_{12}$, yields a $d \times d_3$ and $(d_1 \times d_2 - d) \times d_3$ dimensional square matrix, respectively.

It is possible to fuse two matrices $R_{(12)3}^{(i)}(\lambda)$ to obtain a matrix $R_{(12)(34)}^{(i)}(\lambda)$. Setting $\mu = \rho$ in (7) and multiplying by the projection operators yields two equations similar to (3), (4). This leads to the following definitions:

$$R_{(12)(34)}^{(1)}(\lambda) = \pi_{34}^{(1)} R_{(12)4}^{(1)}(\lambda + \rho) R_{(12)3}^{(1)}(\lambda) \pi_{34}^{(1)} \tag{8}$$

$$R_{(12)(34)}^{(2)}(\lambda) = \pi_{34}^{(2)} R_{(12)4}^{(2)}(\lambda + \rho) R_{(12)3}^{(2)}(\lambda) \pi_{34}^{(2)}. \tag{9}$$

Using the same methods as above one can easily show that these matrices satisfy two Yang–Baxter equations ($i = 1, 2$):

$$R_{(12)(34)}^{(i)}(\lambda - \mu) R_{(12)5}^{(i)}(\lambda) R_{(34)5}^{(i)}(\mu) = R_{(34)5}^{(i)}(\mu) R_{(12)5}^{(i)}(\lambda) R_{(12)(34)}^{(i)}(\lambda - \mu) \tag{10}$$

$$R_{(12)(34)}^{(i)}(\lambda - \mu) R_{(12)(56)}^{(i)}(\lambda) R_{(34)(56)}^{(i)}(\mu) = R_{(34)(56)}^{(i)}(\mu) R_{(12)(56)}^{(i)}(\lambda) R_{(12)(34)}^{(i)}(\lambda - \mu). \tag{11}$$

Again $S_{34}^{-1} R_{(12)(34)}^{(i)}(\lambda) S_{34}$, $i = 1, 2$, still satisfy the above YBE’s. Removing a certain number of vanishing columns and rows by hand, one obtains matrices with smaller dimensions.

Assume now that the original R -matrix is regular and unitary, i.e.

$$R_{12}(0) = c \mathcal{P}_{12} \quad R_{12}(\lambda) R_{21}(-\lambda) = f(\lambda) \mathbb{I} \tag{12}$$

where \mathcal{P} is the permutation operator. The function $f(\lambda)$ is then even, and c is an arbitrary non-vanishing complex number. The fused matrices (8), (9) inherit this property. It is, however, necessary to correctly normalize them for the corresponding value of c not to vanish. This is achieved by the following normalization: redefine $R_{(12)(34)}^{(i)}(\lambda)$ as the right-hand side of (8), (9) multiplied by $(f(\lambda + \rho))^{-1}$. The reason for this normalization is the vanishing of $f(\rho)$, for otherwise the projector $R_{12}(\rho)$ would be invertible and therefore equal to the trivial identity projector \mathbb{I} .

The limit $\lambda \rightarrow 0$ for the redefined matrices (8)–(9) has to be taken with care. Using the YBE one arrives at

$$R_{(12)(34)}^{(1)}(0) = -c^2 \mathcal{P}_{13} \mathcal{P}_{24} S_{12}^{-1} \pi_{12}^{(1)} S_{12} S_{34}^{-1} \pi_{34}^{(1)} S_{34} \tag{13}$$

$$R_{(12)(34)}^{(2)}(0) = +c^2 \mathcal{P}_{13} \mathcal{P}_{24} S_{12}^{-1} \pi_{12}^{(2)} S_{12} S_{34}^{-1} \pi_{34}^{(2)} S_{34} \tag{14}$$

$$R_{(12)(34)}^{(i)}(\lambda) R_{(34)(12)}^{(i)}(-\lambda) = [f(\lambda)]^2 S_{12}^{-1} \pi_{12}^{(i)} S_{12} S_{34}^{-1} \pi_{34}^{(i)} S_{34} \quad i = 1, 2 \tag{15}$$

where $R_{(34)(12)}^{(i)}(\lambda) = \mathcal{P}_{13} \mathcal{P}_{24} R_{(12)(34)}^{(i)}(\lambda) \mathcal{P}_{13} \mathcal{P}_{24}$.

Having obtained regular solutions of the YBE, the usual procedure for constructing integrable spin-chain Hamiltonians with short-range interaction is by taking logarithmic derivatives of the transfer matrices at $\lambda = 0$. The quadratic Hamiltonian density of such integrable hierarchies is nothing but the derivative at $\lambda = 0$ of the matrix $\check{R}(\lambda) = \mathcal{P}R(\lambda)$. Inclusion of the normalizing factor gives

$$\begin{aligned} \check{R}_{(12)(34)}^{(i)}(\lambda) &= \frac{1}{f(\lambda + \rho)} S_{12}^{-1} S_{34}^{-1} \pi_{12}^{(i)} \pi_{34}^{(i)} R_{32}(\lambda + \rho) \check{R}_{13}(\lambda) \\ &\quad \times \check{R}_{24}(\lambda) R_{23}(\lambda - \rho) \pi_{12}^{(i)} \pi_{34}^{(i)} S_{12} S_{34} \quad i = 1, 2. \end{aligned} \tag{16}$$

Taking the limit $\lambda \rightarrow 0$, I find

$$\begin{aligned} \frac{d}{d\lambda} \check{R}_{(12)(34)}^{(i)}(\lambda)_0 &= -(\mp c^2) \frac{f''(0)}{2f'(0)} S_{12}^{-1} \pi_{12}^{(i)} S_{12} S_{34}^{-1} \pi_{34}^{(i)} S_{34} \\ &\quad + \frac{1}{2f'(0)} S_{12}^{-1} S_{34}^{-1} \pi_{12}^{(i)} \pi_{34}^{(i)} \frac{d^2}{d\lambda^2} (R_{32}(\lambda + \rho) \check{R}_{13}(\lambda) \\ &\quad \times \check{R}_{24}(\lambda) R_{23}(\lambda - \rho))_0 \pi_{12}^{(i)} \pi_{34}^{(i)} S_{12} S_{34}. \end{aligned} \tag{17}$$

The two signs are as in the regularity equations above. The first term is proportional to the identity and may be dropped.

It is in fact possible to fuse an arbitrary product of R matrices to obtain solutions of the YBE corresponding to most representations of a given Lie algebra. This was carried out in detail for $sl(2)$ and $sl(3)$ in [13, 14], and also works for the other algebras [15].

The above fusing scheme is now shown to work for the multiplicity A_{m-1} models.

3. New integrable models

As seen above, apart from satisfying the Yang–Baxter equation, the projector property is the only additional ingredient needed to construct fused matrices. In particular, R does not have to correspond to a smallest representation to be able to apply the fusion method. The degeneration of the generically invertible R -matrix to a projector, for a certain value of the spectral parameter, is not automatic. However, it is a quite common property, especially for the R -matrices based on Lie algebras. For the models studied here, a quantum group structure does not seem to exist, and one has to verify that a projector point exists.

I now apply this formalism to the models of [1]. This will yield new matrices which define new integrable hierarchies. The models of [1] are defined as follows. Take positive integers n_i and m such that

$$\sum_{i=1}^m n_i = n \quad \text{and} \quad 1 \leq n_1 \leq \dots \leq n_m \leq n-1. \quad (18)$$

The inequality restrictions avoid multiple counting of models, but can otherwise be relaxed. The set of n basis states is the disjoint union of m sets \mathcal{A}_i :

$$\text{card}(\mathcal{A}_i) = n_i \quad \mathcal{A}_i \cap \mathcal{A}_j = \emptyset \quad \text{for } i \neq j. \quad (19)$$

\mathcal{A}_i should not be confused with the Lie algebra $sl(i+1)$. Let $E^{\alpha\beta}$ be a square matrix with a one at row α and column β and zeros otherwise. Define the following operators:

$$\tilde{P}^{(+)} = \sum_{1 \leq i < j \leq m} \sum_{\alpha_i \in \mathcal{A}_i} \sum_{\alpha_j \in \mathcal{A}_j} E^{\alpha_j \alpha_i} \otimes E^{\alpha_i \alpha_j} \quad (20)$$

$$\tilde{P}^{(-)} = \sum_{1 \leq i < j \leq m} \sum_{\alpha_i \in \mathcal{A}_i} \sum_{\alpha_j \in \mathcal{A}_j} E^{\alpha_i \alpha_j} \otimes E^{\alpha_j \alpha_i} \quad (21)$$

$$\tilde{P}^{(3)} = \sum_{1 \leq i < j \leq m} \sum_{\alpha_i \in \mathcal{A}_i} \sum_{\alpha_j \in \mathcal{A}_j} (x E^{\alpha_j \alpha_j} \otimes E^{\alpha_i \alpha_i} + x^{-1} E^{\alpha_i \alpha_i} \otimes E^{\alpha_j \alpha_j}). \quad (22)$$

The twist parameters were taken equal to one arbitrary complex parameter x . Let $y = e^{i\lambda}$ and $q = e^{i\gamma}$, where λ is the spectral parameter and γ the quantum parameter. The R -matrix is then given by

$$R(\lambda) = \mathcal{P} \sin(\lambda + \gamma) + \tilde{P} \sin \lambda \quad (23)$$

$$\tilde{P} \equiv \tilde{P}^{(3)} - (q^{-1} \tilde{P}^{(+)} + q \tilde{P}^{(-)}). \quad (24)$$

This model is denoted by $(n_1, \dots, n_m; m, n)$. For $n = m$ and $x = 1$ one obtains the A_{m-1} R -matrix of [5]. $R(\lambda)$ is regular and unitary:

$$R_{12}(0) = \mathcal{P}_{12} \sin \gamma \quad (25)$$

$$R_{12}(\lambda) R_{21}(-\lambda) = \mathbb{I} f(\lambda) = \mathbb{I} \sin(\gamma + \lambda) \sin(\gamma - \lambda) \quad (26)$$

where $R_{21}(\lambda) = \mathcal{P}_{12} R_{12}(\lambda) \mathcal{P}_{12}$. As seen in section 2, these properties are inherited by some of the fused matrices.

The right-hand side of (26) shows that $\lambda = \pm\gamma$ are possible projector points. Further checks show that matrix (23) yields a projector at $\lambda = \rho = -\gamma$: $R(-\gamma) = -\tilde{P} \sin \gamma$. Let

$$\pi^{(1)} = \frac{1}{x+x^{-1}} \tilde{P} \quad \pi^{(2)} = \mathbb{I} - \pi^{(1)}. \tag{27}$$

One has $(\pi^{(i)})^2 = \pi^{(i)}$, $i = 1, 2$, and $\pi^{(1)}\pi^{(2)} = \pi^{(2)}\pi^{(1)} = 0$. The dimensions of these projectors are given by their traces:

$$\text{tr}(\pi^{(1)}) = \sum_{i < j} n_i n_j = \frac{1}{2} \left(n^2 - \sum_i (n_i)^2 \right) \tag{28}$$

$$\text{tr}(\pi^{(2)}) = n^2 - \sum_{i < j} n_i n_j = \frac{1}{2} \left(n^2 + \sum_i (n_i)^2 \right). \tag{29}$$

The matrix S which diagonalizes $\pi^{(1)}$ and $\pi^{(2)}$ is given by

$$\begin{aligned} S = & \sum_{i=1}^m \sum_{\alpha_i \in \mathcal{A}_i} \sum_{\beta_i \in \mathcal{A}_i} E^{\alpha_i \alpha_i} \otimes E^{\beta_i \beta_i} \\ & + \sum_{1 \leq i < j \leq m} \sum_{\alpha_i \in \mathcal{A}_i} \sum_{\alpha_j \in \mathcal{A}_j} \left(E^{\alpha_i \alpha_i} \otimes E^{\alpha_j \alpha_j} - \frac{x}{q} E^{\alpha_j \alpha_j} \otimes E^{\alpha_i \alpha_i} \right) \\ & + \sum_{1 \leq i < j \leq m} \sum_{\alpha_i \in \mathcal{A}_i} \sum_{\alpha_j \in \mathcal{A}_j} \left(E^{\alpha_i \alpha_j} \otimes E^{\alpha_j \alpha_i} + \frac{1}{xq} E^{\alpha_j \alpha_i} \otimes E^{\alpha_i \alpha_j} \right) \end{aligned} \tag{30}$$

$$\begin{aligned} S^{-1} = & \sum_{i=1}^m \sum_{\alpha_i \in \mathcal{A}_i} \sum_{\beta_i \in \mathcal{A}_i} E^{\alpha_i \alpha_i} \otimes E^{\beta_i \beta_i} \\ & + \frac{q}{x+x^{-1}} \sum_{1 \leq i < j \leq m} \sum_{\alpha_i \in \mathcal{A}_i} \sum_{\alpha_j \in \mathcal{A}_j} \left(\frac{x}{q} E^{\alpha_i \alpha_i} \otimes E^{\alpha_j \alpha_j} - E^{\alpha_j \alpha_j} \otimes E^{\alpha_i \alpha_i} \right) \\ & + \frac{q}{x+x^{-1}} \sum_{1 \leq i < j \leq m} \sum_{\alpha_i \in \mathcal{A}_i} \sum_{\alpha_j \in \mathcal{A}_j} \left(E^{\alpha_i \alpha_j} \otimes E^{\alpha_j \alpha_i} + \frac{1}{xq} E^{\alpha_j \alpha_i} \otimes E^{\alpha_i \alpha_j} \right). \end{aligned} \tag{31}$$

The diagonalized projectors then read

$$S^{-1} \pi^{(1)} S = \sum_{1 \leq i < j \leq m} \sum_{\alpha_i \in \mathcal{A}_i} \sum_{\alpha_j \in \mathcal{A}_j} E^{\alpha_j \alpha_j} \otimes E^{\alpha_i \alpha_i} \tag{32}$$

$$S^{-1} \pi^{(2)} S = \sum_{i=1}^m \sum_{\alpha_i \in \mathcal{A}_i} \sum_{\beta_i \in \mathcal{A}_i} E^{\alpha_i \alpha_i} \otimes E^{\beta_i \beta_i} + \sum_{1 \leq i < j \leq m} \sum_{\alpha_i \in \mathcal{A}_i} \sum_{\alpha_j \in \mathcal{A}_j} E^{\alpha_i \alpha_i} \otimes E^{\alpha_j \alpha_j}. \tag{33}$$

The explicit expressions of the four fused matrices $R_{(12)3}^{(i)}$ and $R_{(12)\{34\}}^{(i)}$ can be found by straightforward if tedious expansions of the products in (5), (6), (8), (9), using the explicit expressions (23), (27), (30), (31). Similarly, the quadratic Hamiltonian density is obtained by replacing the matrices in (17) and expanding the products.

The following point is worth mentioning. Since $R_{12}(-\gamma) R_{21}(\gamma) = 0$, it may seem that the complementary projector $\pi^{(2)}$ is proportional to $R_{21}(\gamma)$. However, this is *not* the case generically. It is true only when all the n_i are equal to one and $x = q^{\pm 1}$. This is another distinguishing feature of the $n_i \neq 1$ models.

4. Some $sl(2)$ models

Consider now the $m = 2$ case, i.e. the XXC models [17]. They have an underlying $sl(2)$ structure. Their R matrix is just the multistate version of the one corresponding to the spin- $\frac{1}{2}$ model ($n_1 = n_2 = 1$).

Tedious but simple calculations lead to the following matrix which carries spin-0 \times spin- $\frac{1}{2}$:

$$R_{(12)3}^{(1)}(\lambda) = \sin \lambda \sin(\lambda + 2\gamma) \left(\sum_{\alpha_1, \beta_1} \sum_{\alpha_2} x E^{\alpha_2 \alpha_2} \otimes E^{\beta_1 \alpha_1} \otimes E^{\alpha_1 \beta_1} \right. \\ \left. + \sum_{\alpha_1} \sum_{\alpha_2, \beta_2} x^{-1} E^{\beta_2 \alpha_2} \otimes E^{\alpha_1 \alpha_1} \otimes E^{\alpha_2 \beta_2} \right). \quad (34)$$

The dimension of this matrix is $n_1 n_2 n$, as expected. The corresponding two-dimensional vertex model has $n_1 n_2$ possible states on, e.g., the horizontal links and n states on the vertical links.

Note that the $x + x^{-1}$ denominator has dropped out of the final result and therefore R is defined for all finite values of x , including $x = \pm i$. This also holds for the three fused matrices given below.

The matrix which carries spin-0 \times spin-0 can then be obtained:

$$R_{(12)(34)}^{(1)}(\lambda) = \frac{\sin(\lambda - \gamma) \sin(\lambda + \gamma) \sin(\lambda + 2\gamma)}{\sin(2\gamma - \lambda)} \sum_{\alpha_1, \beta_1} \sum_{\alpha_2, \beta_2} E^{\alpha_2 \beta_2} \otimes E^{\alpha_1 \beta_1} \otimes E^{\beta_2 \alpha_2} E^{\beta_1 \alpha_1} \\ = \frac{\sin(\lambda - \gamma) \sin(\lambda + \gamma) \sin(\lambda + 2\gamma)}{\sin(2\gamma - \lambda)} \mathcal{P}_{13}^{(A_2)} \mathcal{P}_{24}^{(A_1)}. \quad (35)$$

Here $\mathcal{P}_{13}^{(A_2)}$ and $\mathcal{P}_{24}^{(A_1)}$ are the permutation operators in the subspaces \mathcal{A}_2 and \mathcal{A}_1 , respectively. The dimension of this matrix is $(n_1 n_2)^2$, as it should be. It satisfies the regularity and unitarity properties (13), (15). Here the vertex model has $n_1 n_2$ possible states on both the horizontal and vertical links. The resulting spin-chain is however rather trivial as the λ -dependence can be normalized away, and the operator part yields $S_{12}^{-1} \pi_{12}^{(1)} S_{12} S_{34}^{-1} \pi_{34}^{(1)} S_{34}$, i.e. the identity operators in the spin-0 subspaces.

The matrix carrying spin-1 \times spin- $\frac{1}{2}$ is

$$\frac{1}{\sin(\lambda + \gamma)} R_{(12)3}^{(2)}(\lambda) = \sin(\lambda + 2\gamma) \sum_{\alpha_1, \beta_1, \gamma_1} E^{\alpha_1 \beta_1} \otimes E^{\gamma_1 \alpha_1} \otimes E^{\beta_1 \gamma_1} \\ + \sin(\lambda + 2\gamma) \sum_{\alpha_2, \beta_2, \gamma_2} E^{\alpha_2 \beta_2} \otimes E^{\gamma_2 \alpha_2} \otimes E^{\beta_2 \gamma_2} \\ + y \sin(2\gamma) \sum_{\alpha_1} \sum_{\alpha_2, \beta_2} E^{\alpha_2 \alpha_1} \otimes E^{\beta_2 \alpha_2} \otimes E^{\alpha_1 \beta_2} \\ + q y \sin \gamma \sum_{\alpha_1, \beta_1} \sum_{\alpha_2} E^{\beta_1 \alpha_1} \otimes E^{\alpha_2 \beta_1} \otimes E^{\alpha_1 \alpha_2} \\ + x \sin(\lambda + \gamma) \sum_{\alpha_1, \beta_1} \sum_{\alpha_2} E^{\beta_1 \alpha_1} \otimes E^{\alpha_2 \alpha_2} \otimes E^{\alpha_1 \beta_1} \\ + x^{-1} \sin(\lambda + \gamma) \sum_{\alpha_1} \sum_{\alpha_2, \beta_2} E^{\alpha_1 \alpha_1} \otimes E^{\beta_2 \alpha_2} \otimes E^{\alpha_2 \beta_2} \\ + x y^{-1} \sin \gamma \sum_{\alpha_1} \sum_{\alpha_2, \beta_2} E^{\alpha_1 \alpha_2} \otimes E^{\beta_2 \beta_2} \otimes E^{\alpha_2 \alpha_1}$$

$$\begin{aligned}
 &+ x^{-1} q^{-1} y^{-1} \sin(2\gamma) \sum_{\alpha_1, \beta_1} \sum_{\alpha_2} E^{\alpha_1 \alpha_1} \otimes E^{\beta_1 \alpha_2} \otimes E^{\alpha_2 \beta_1} \\
 &+ x^2 \sin \lambda \sum_{\alpha_1} \sum_{\alpha_2, \beta_2} E^{\alpha_2 \alpha_2} \otimes E^{\beta_2 \beta_2} \otimes E^{\alpha_1 \alpha_1} \\
 &+ x^{-2} \sin \lambda \sum_{\alpha_1, \beta_1} \sum_{\alpha_2} E^{\alpha_1 \alpha_1} \otimes E^{\beta_1 \beta_1} \otimes E^{\alpha_2 \alpha_2}. \tag{36}
 \end{aligned}$$

The vertex model model has $(n_1)^2 + (n_2)^2 + n_1 n_2$ states on the horizontal links and n states on the vertical links.

The preceding matrix is then used to find the spin-1 \times spin-1 matrix:

$$\begin{aligned}
 &\frac{\sin(2\gamma - \lambda)}{\sin(\lambda + \gamma)} R_{(12)(34)}^{(2)}(\lambda) \\
 &= \sin(\lambda + \gamma) \sin(\lambda + 2\gamma) \sum_{\alpha_1, \beta_1, \gamma_1, \delta_1} E^{\alpha_1 \beta_1} \otimes E^{\gamma_1 \delta_1} \otimes E^{\beta_1 \alpha_1} \otimes E^{\delta_1 \gamma_1} \\
 &+ \sin(\lambda + \gamma) \sin(\lambda + 2\gamma) \sum_{\alpha_2, \beta_2, \gamma_2, \delta_2} E^{\alpha_2 \beta_2} \otimes E^{\gamma_2 \delta_2} \otimes E^{\beta_2 \alpha_2} \otimes E^{\delta_2 \gamma_2} \\
 &+ y \sin(2\gamma) \sin(\lambda + \gamma) \sum_{\alpha_1} \sum_{\alpha_2, \beta_2, \gamma_2} E^{\alpha_2 \alpha_1} \otimes E^{\beta_2 \gamma_2} \otimes E^{\alpha_1 \alpha_2} \otimes E^{\gamma_2 \beta_2} \\
 &+ y^{-1} \sin(2\gamma) \sin(\lambda + \gamma) \sum_{\alpha_1, \beta_1, \gamma_1} \sum_{\alpha_2} E^{\alpha_1 \beta_1} \otimes E^{\gamma_1 \alpha_2} \otimes E^{\beta_1 \alpha_1} \otimes E^{\alpha_2 \gamma_1} \\
 &+ y \sin(2\gamma) \sin(\lambda + \gamma) \sum_{\alpha_1, \beta_1, \gamma_1} \sum_{\alpha_2} E^{\alpha_1 \beta_1} \otimes E^{\alpha_2 \gamma_1} \otimes E^{\beta_1 \alpha_1} \otimes E^{\gamma_1 \alpha_2} \\
 &+ y^{-1} \sin(2\gamma) \sin(\lambda + \gamma) \sum_{\alpha_1} \sum_{\alpha_2, \beta_2, \gamma_2} E^{\alpha_1 \alpha_2} \otimes E^{\beta_2 \gamma_2} \otimes E^{\alpha_2 \alpha_1} \otimes E^{\gamma_2 \beta_2} \\
 &+ (\sin \gamma \sin(2\gamma) + \sin \lambda \sin(\lambda + \gamma)) \sum_{\alpha_1, \beta_1} \sum_{\alpha_2, \beta_2} E^{\alpha_1 \beta_1} \otimes E^{\alpha_2 \beta_2} \otimes E^{\beta_1 \alpha_1} \otimes E^{\beta_2 \alpha_2} \\
 &+ y^2 \sin \gamma \sin(2\gamma) \sum_{\alpha_1, \beta_1} \sum_{\alpha_2, \beta_2} E^{\alpha_2 \alpha_1} \otimes E^{\beta_2 \beta_1} \otimes E^{\alpha_1 \alpha_2} \otimes E^{\beta_1 \beta_2} \\
 &+ y^{-2} \sin \gamma \sin(2\gamma) \sum_{\alpha_1, \beta_1} \sum_{\alpha_2, \beta_2} E^{\alpha_1 \alpha_2} \otimes E^{\beta_1 \beta_2} \otimes E^{\alpha_2 \alpha_1} \otimes E^{\beta_2 \beta_1} \\
 &+ 2x q^{-1} y \cos \gamma \sin(2\gamma) \sin \lambda \sum_{\alpha_1, \beta_1} \sum_{\alpha_2, \beta_2} E^{\alpha_2 \alpha_1} \otimes E^{\beta_2 \alpha_2} \otimes E^{\alpha_1 \beta_1} \otimes E^{\beta_1 \beta_2} \\
 &+ x^{-1} q y \sin \gamma \sin \lambda \sum_{\alpha_1, \beta_1} \sum_{\alpha_2, \beta_2} E^{\alpha_1 \beta_1} \otimes E^{\alpha_2 \alpha_1} \otimes E^{\beta_1 \beta_2} \otimes E^{\beta_2 \alpha_2} \\
 &+ x^2 \sin \lambda \sin(\lambda + \gamma) \sum_{\alpha_1, \beta_1, \gamma_1} \sum_{\alpha_2} E^{\alpha_1 \beta_1} \otimes E^{\alpha_2 \alpha_2} \otimes E^{\beta_1 \gamma_1} \otimes E^{\gamma_1 \alpha_1} \\
 &+ x^2 \sin \lambda \sin(\lambda + \gamma) \sum_{\alpha_1} \sum_{\alpha_2, \beta_2, \gamma_2} E^{\alpha_2 \beta_2} \otimes E^{\gamma_2 \alpha_2} \otimes E^{\alpha_1 \alpha_1} \otimes E^{\beta_2 \gamma_2} \\
 &+ x^{-2} \sin \lambda \sin(\lambda + \gamma) \sum_{\alpha_1, \beta_1, \gamma_1} \sum_{\alpha_2} E^{\alpha_1 \beta_1} \otimes E^{\gamma_1 \alpha_1} \otimes E^{\beta_1 \gamma_1} \otimes E^{\alpha_2 \alpha_2} \\
 &+ x^{-2} \sin \lambda \sin(\lambda + \gamma) \sum_{\alpha_1} \sum_{\alpha_2, \beta_2, \gamma_2} E^{\alpha_1 \alpha_1} \otimes E^{\alpha_2 \beta_2} \otimes E^{\beta_2 \gamma_2} \otimes E^{\gamma_2 \alpha_2}
 \end{aligned}$$

$$\begin{aligned}
& + x^3 q y^{-1} \sin \gamma \sin \lambda \sum_{\alpha_1, \beta_1} \sum_{\alpha_2, \beta_2} E^{\alpha_1 \alpha_2} \otimes E^{\beta_2 \beta_2} \otimes E^{\beta_1 \beta_1} \otimes E^{\alpha_2 \alpha_1} \\
& + 2x^{-3} q^{-1} y^{-1} \cos \gamma \sin(2\gamma) \sin \lambda \sum_{\alpha_1, \beta_1} \sum_{\alpha_2, \beta_2} E^{\alpha_1 \alpha_1} \otimes E^{\beta_1 \alpha_2} \otimes E^{\alpha_2 \beta_1} \otimes E^{\beta_2 \beta_2} \\
& + x^4 \sin(\lambda - \gamma) \sin \lambda \sum_{\alpha_1, \beta_1} \sum_{\alpha_2, \beta_2} E^{\alpha_2 \alpha_2} \otimes E^{\beta_2 \beta_2} \otimes E^{\alpha_1 \alpha_1} \otimes E^{\beta_1 \beta_1} \\
& + x^{-4} \sin(\lambda - \gamma) \sin \lambda \sum_{\alpha_1, \beta_1} \sum_{\alpha_2, \beta_2} E^{\alpha_1 \alpha_1} \otimes E^{\beta_1 \beta_1} \otimes E^{\alpha_2 \alpha_2} \otimes E^{\beta_2 \beta_2}. \quad (37)
\end{aligned}$$

This matrix does satisfy the regularity and unitarity properties (14), (15). The number of states on both the horizontal and vertical links is now $(n_1)^2 + (n_2)^2 + n_1 n_2$.

The four matrices (34), (35), (36), (37) have 2, 1, 10, 19 types of terms, respectively. This was expected from the S^z conservation of $sl(2)$. When $n_1 = n_2 = 1$ the matrices give the $sl(2)$ 2-, 1-, 10- and 19-vertex models. The latter two models can be found in [18–20]. To carry out a comparison it is necessary to relabel the states so that R looks like an operator acting on the tensor product of two spaces: $R = \sum E \otimes E$. For instance, in (36), with $\mathcal{A}_1 = \{1\}$, $\mathcal{A}_2 = \{2\}$, one can take the matrix element $\sin(\lambda + \gamma) \sin(\lambda + 2\gamma) E^{11} \otimes E^{11} \otimes E^{11}$ to correspond to $|+1\rangle|+\frac{1}{2}\rangle \times |+1\rangle|+\frac{1}{2}\rangle$, and $(x q y)^{-1} \sin(2\gamma) \sin(\lambda + \gamma) E^{11} \otimes E^{12} \otimes E^{21}$ to $|+1\rangle|-\frac{1}{2}\rangle \times |0\rangle|+\frac{1}{2}\rangle$. The matrix (19) of [20] can be obtained by fusing the symmetric form of the $sl(2)$ R -matrix [17]. It is also necessary to do a gauge transformation after fusing, in order to render the resulting matrix completely symmetric. This introduces the unusual square-root element d of that reference. Finally, let $u = \lambda + \eta/2$ and $\eta = \gamma$ to complete the identification. Similar remarks apply to the identification of the 19-vertex models.

The interpretation of the multiple-states in terms of $sl(2)$ states is simple. Recall first that for the models (23) with $m = 2$, the interpretation is

$$\text{state } (\alpha_1) \longleftrightarrow |+\frac{1}{2}\rangle_{\alpha_1} \quad \text{state } (\alpha_2) \longleftrightarrow |-\frac{1}{2}\rangle_{\alpha_2}. \quad (38)$$

This yields the following identifications when two spin- $\frac{1}{2}$ spaces are fused:

$$\text{state } (\alpha_2 \alpha_1) \longleftrightarrow |0\rangle_{\alpha_2 \alpha_1} \quad n_1 n_2 \quad (39)$$

for the spin-0 representation obtained from $\pi^{(1)}$, and

$$\text{state } (\alpha_1 \beta_1) \longleftrightarrow |+1\rangle_{\alpha_1 \beta_1} \quad (n_1)^2 \quad (40)$$

$$\text{state } (\alpha_1 \alpha_2) \longleftrightarrow |0\rangle_{\alpha_1 \alpha_2} \quad n_1 n_2 \quad (41)$$

$$\text{state } (\alpha_2 \beta_2) \longleftrightarrow |-1\rangle_{\alpha_2 \beta_2} \quad (n_2)^2 \quad (42)$$

for the spin-1 representation obtained from $\pi^{(2)}$. The numbers on the right are the number of states of the corresponding type. Whereas the numbers of copies of the states $|\pm 1\rangle$ are uncorrelated, $(n_1)^2$ and $(n_2)^2$, the number of states of type $|0\rangle$ has to be $n_1 n_2$. Thus a naive trial to obtain the multistate version of the spin-1 \times spin-1 model would have failed. Fusion gives the correct answer.

The first derivative at $\lambda = 0$ of $\check{R}_{(12)(34)}^{(2)}(\lambda) = \mathcal{P}_{13} \mathcal{P}_{24} R_{(12)(34)}^{(2)}(\lambda)$, for the matrix of (37), gives the Hamiltonian density H_{i+1} for the multistate version of the spin-1 model. The $n_1 = n_2 = 1$ Hamiltonian is also known as the Fateev–Zamolodchikov model [18].

5. Concluding remarks

The multiplicity A_{m-1} models were shown to allow fusion and multistate models were obtained for higher-dimensional representations. The spin-0 and spin-1 models were derived explicitly. This provided in particular the generalization of the Fateev–Zamolodchikov model.

The underlying $sl(m)$ structure of these models was justified in [1]. It provides a natural way to label them in terms of $sl(m)$ representations. This language has been used throughout the paper and in particular in section 4. I recall here the main points in view of an extension to the higher-dimensional models just obtained. The matrices (23) have the following structure: $R(\lambda) = \sum_k g_k(\lambda) \mathcal{O}_k$, where all the spectral parameter dependence is contained in the functions $g_k(\lambda)$. The operators \mathcal{O}_k may depend on the other parameters. Replacing R in the YBE's it is required to satisfy, and identifying the coefficients of the linearly independent functions yield a set of equations for the operators \mathcal{O}_k , where the spectral parameters do not appear. In the case of the \check{O}_k of (23) one has simple n_i -independent trigonometric functions, and obtains the Hecke algebra, for all values of the n_i 's, and in particular for $n_i = 1$. This and the fact that the operators have the same structure as for $n_i = 1$ shows that one has the natural generalization. The algebraic Bethe Ansatz was also found to be based on the Dynkin diagram of A_{m-1} . Thus the functional and operatorial structures of $R(\lambda) = \sum_k g_k(\lambda) \mathcal{O}_k$ are the same as for $n_i = 1$. The quite general fusion construction reviewed in section 2 preserves the functional and algebraic structures. Therefore one again has the natural multistate generalization for the higher-dimensional representations.

The rational limit of all the matrices considered above exist and provide rational solutions to the various Yang–Baxter equations. This limit is obtained by rescaling λ to $\gamma\lambda$, and dividing by an appropriate power of γ or $\sin \gamma$ in order to obtain a finite limit.

The models (23) have extended symmetries [1]: $sl(n_1) \oplus \dots \oplus sl(n_m) \oplus u(1) \oplus \dots \oplus u(1)$. It is obvious that the higher-dimensional models inherit similar symmetries. These are related to the existence of many states of a same type, as seen in the spin-0 and spin-1 examples. Most (but not all) diagonal operators commute with the R -matrix, as they either correspond to $u(1)$ charges of states of the same type, or to realizations of S^z . The non-diagonal operators which commute with the R -matrix are those which acts within the space spanned by a state and its copies.

These symmetries will also be reflected in a diagonalization by Bethe ansatz, as happened for the fundamental models [1]. The diagonalization makes use of the standard techniques associated with higher-dimensional representations. In particular, it uses fusion to find relations between the various transfer matrices.

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